

A LOCAL WEIGHTED AXLER-ZHENG THEOREM IN \mathbb{C}^n

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ABSTRACT. We prove a local weighted version of Axler-Zheng's Theorem on C^2 -smooth bounded pseudoconvex domains in \mathbb{C}^n .

1. INTRODUCTION

1.1. History. In the theory of Bergman space operators on the open unit disk \mathbb{D} , Axler-Zheng theorem [AZ98] provides an important characterization of compactness of a large class of operators in terms of their Berezin transforms. Specifically this theorem states that if S is a finite sum of finite products of Toeplitz operators on the Bergman space $A^2(\mathbb{D})$ whose symbols are in $L^\infty(\mathbb{D})$, then S is compact if and only if the Berezin transform of S , $BS(z) \rightarrow 0$ as $|z| \rightarrow 1$. This theorem has been extended by Suarez [Suá07] to include all operators in the Toeplitz algebra in the unit ball in \mathbb{C}^n . Englis [Eng99] extended the Axler-Zheng theorem to irreducible bounded symmetric domains and the unit polydisk. Mitkovski, Suarez and Wick [MSW13] proved a weighted version of Suarez's result on the unit ball in \mathbb{C}^n . Using the techniques of several complex variables, Čučković and Şahutoğlu [ČŞ13] proved a version of the Axler-Zheng theorem on smooth bounded pseudoconvex domains on which the $\bar{\partial}$ -Neumann operator is compact. The use of the $\bar{\partial}$ techniques required that the operators in their theorem belong to the algebra $\mathcal{T}(\bar{\Omega})$ which is the norm closed algebra generated by $\{T_\phi : \phi \in C(\bar{\Omega})\}$. Recently, in her Master's thesis [Kre14], Kreutzer generalized Čučković and Şahutoğlu's result in a more abstract setting.

In this paper our aim is to extend the previous result of Čučković and Şahutoğlu in two ways: Firstly, we want to remove the hypothesis of the compactness of the $\bar{\partial}$ -Neumann operator on Ω . We also want to consider weighted Bergman spaces. Our main theorem gives a local version of the Axler-Zheng theorem for a wide class of domains in \mathbb{C}^n . The novelty of our approach is to use the inflation of the domain argument pioneered by Forelli-Rudin and Ligocka [FR75, Lig89]. The second important ingredient is the B-regularity of the inflated domain which will give us the compactness of $\bar{\partial}$, thus replacing the assumption on the compactness of the $\bar{\partial}$ -Neumann operator. As a corollary we obtain a weighted version of the Axler-Zheng theorem for strongly pseudoconvex domains, which itself is a new result.

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1.2. Preliminaries. Let Ω be a C^1 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ . We denote the boundary of Ω by $b\Omega$. Let $L^2(\Omega, (-\rho)^r)$ denote the square integrable functions on Ω with respect to the measure $(-\rho)^r dV$ where dV denotes the Lebesgue measure, $r \geq 0$, and

$$A^2(\Omega, (-\rho)^r) = \left\{ f \in L^2(\Omega, (-\rho)^r) : f \text{ is holomorphic} \right\}.$$

Since $A^2(\Omega, (-\rho)^r)$ is a closed subspace of $L^2(\Omega, (-\rho)^r)$ a bounded orthogonal projection

$$P_r : L^2(\Omega, (-\rho)^r) \rightarrow A^2(\Omega, (-\rho)^r),$$

(called Bergman projection) exists. P_r is an integral operator of the form

$$P_r(f)(z) = \int_{\Omega} K^r(z, \zeta) f(\zeta) (-\rho)^r dV$$

for $f \in L^2(\Omega, (-\rho)^r)$. The integral kernel $K^r(z, \zeta)$ is called the Bergman kernel and the normalized Bergman kernel $k_z^r(\zeta)$ is defined as $k_z^r(\zeta) = \frac{K^r(\zeta, z)}{\sqrt{K^r(z, z)}}$. When $r = 0$ we drop the superscript r ; that is, $K = K_{\Omega}$ denotes the unweighted Bergman kernel and k_z denotes the unweighted normalized Bergman kernel. For a bounded operator T on $A^2(\Omega, (-\rho)^r)$, the Berezin transform $B_r T$ of T is defined as

$$B_r T(z) = \langle T k_z^r, k_z^r \rangle_r$$

where and $\langle \cdot, \cdot \rangle_r$ is the inner product on $A^2(\Omega, (-\rho)^r)$.

For $\phi \in L^{\infty}(\Omega)$, the weighted Toeplitz operator T_{ϕ}^r and the weighted Hankel operator H_{ϕ}^r are defined as follows

$$T_{\phi}^r = P^r M_{\phi}$$

$$H_{\phi}^r = (I - P^r) M_{\phi}$$

where $M_{\phi} : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$ denotes the multiplication by ϕ .

We use $\mathcal{T}(\overline{\Omega}, (-\rho)^r)$ to denote the norm closed subalgebra of bounded linear operators on $A^2(\Omega, (-\rho)^r)$ generated by the set of Toeplitz operators $\{T_{\phi}^r : \phi \in C(\overline{\Omega})\}$. For $\phi \in L^{\infty}$ we define $B_r \phi = B_r T_{\phi}$.

In this paper we look at weighted Hankel and Toeplitz operators on various domains and various weighted spaces. Whenever we need to clarify where these operators are defined, we will use appropriate subscripts and superscripts. In particular, when we need to emphasize the underlying domain we will write $P^{\Omega}, K_{\Omega}(z, \zeta), H_{\phi}^{\Omega}$, and T_{ϕ}^{Ω} , where the Bergman spaces are unweighted. When we have weighted spaces and we need to indicate the domain and the weight we will write $P^{\Omega, r}, K_{\Omega}^r(z, \zeta), H_{\phi}^{\Omega, r}$, and $T_{\phi}^{\Omega, r}$.

1.3. Main Result. We start with the following two definitions that capture the local structure of the main theorem. To motivate the following definition, if $f_j \rightarrow f$ weakly in $A^2(\Omega)$ then for any point $p \in b\Omega$ and $r > 0$ one can show that $f_j \rightarrow f$ weakly in $A^2(\Omega \cap B(p, r))$ where $B(p, r)$ is the open ball centered at p with radius r .

Definition 1. Let $r \geq 0$ and Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ . Furthermore, let $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$ be a sequence and $f \in A^2(\Omega, (-\rho)^r)$. We say that $\{f_j\}$ converges to f *weakly about strongly pseudoconvex points* if

- i. $f_j \rightarrow f$ weakly in $A^2(\Omega, (-\rho)^r)$ as $j \rightarrow \infty$,
- ii. in case Γ_Ω , the set of the weakly pseudoconvex points in $b\Omega$, is non-empty, there exists an open neighborhood U of Γ_Ω such that $\|f_j - f\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0$ as $j \rightarrow \infty$.

We note that on strongly pseudoconvex domains, sequences converging weakly about strongly pseudoconvex points and weakly convergent sequences coincide.

Definition 2. Let r , Ω , and ρ be as above. Furthermore, let $T : A^2(\Omega, (-\rho)^r) \rightarrow A^2(\Omega, (-\rho)^r)$ be a bounded linear operator. We say that T is *compact about strongly pseudoconvex points* if $Tf_j \rightarrow Tf$ in $A^2(\Omega, (-\rho)^r)$ whenever $f_j \rightarrow f$ weakly about strongly pseudoconvex points.

Remark 1. As shown in Proposition 2 below, it is interesting that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

With the help of these two definitions, we state our main result as follows.

Theorem 1. Let r be a nonnegative real number, Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ , and $T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r)$. Then T is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$ if and only if $\lim_{z \rightarrow p} B_r T(z) = 0$ for any strongly pseudoconvex point $p \in b\Omega$.

If Ω is a strongly pseudoconvex domain then we have the following corollary.

Corollary 1. Let r be a nonnegative real number, Ω be a C^2 -smooth bounded strongly pseudoconvex domain in \mathbb{C}^n with a defining function ρ , and $T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r)$. Then T is compact on $A^2(\Omega, (-\rho)^r)$ if and only if $\lim_{z \rightarrow p} B_r T(z) = 0$ for any $p \in b\Omega$.

Remark 2. In the case of the unit ball \mathbb{B}^n in \mathbb{C}^n and $\rho(z) = |z|^2 - 1$, we partially recover [MSW13, Theorem 1.1]. Unlike the arguments on the unit ball, the proof of Corollary 1 does not require any explicit form for the weight or the weighted Bergman kernel.

2. PROOF OF THEOREM 1

In this section, before we prove Theorem 1, we present some propositions and lemmas that encapsulate the technical details of the proof.

Proposition 1. Let Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n and $\{T_j\}$ be a sequence of operators compact about strongly pseudoconvex points that converge to T in the operator norm. Then T is compact about strongly pseudoconvex points.

Proof. Let $\{f_j\}$ be a sequence in $A^2(\Omega, (-\rho)^r)$ that converges to 0 weakly about strongly pseudoconvex points. Since $f_j \rightarrow 0$ weakly there exists $C > 0$ such that

$$\sup\{\|f_j\| : j = 1, 2, 3, \dots\} \leq C.$$

Then for any k we have

$$\|Tf_j\| \leq \|(T - T_k)f_j\| + \|T_kf_j\| \leq C\|T - T_k\| + \|T_kf_j\|.$$

Let $\varepsilon > 0$ be given. Since $T_j \rightarrow T$ in the operator norm, we choose k_ε such that $\|T - T_{k_\varepsilon}\| \leq \varepsilon$. Then

$$\limsup_{j \rightarrow \infty} \|Tf_j\| \leq C\varepsilon + \limsup_{j \rightarrow \infty} \|T_{k_\varepsilon}f_j\| \leq C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $Tf_j \rightarrow 0$. That is, T is compact about strongly pseudoconvex points. \square

One of the key ideas in the proof is to use an inflated domain over Ω to understand the weighted Bergman spaces. For this purpose, unless stated otherwise, for the rest of the paper, Ω will be a bounded pseudoconvex domain in \mathbb{C}^n with C^2 -smooth boundary, ρ will be a defining function for Ω , and

$$(1) \quad \Omega_r^p = \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^p : z \in \Omega \text{ and } \rho(z) + |w_1|^{2p/r} + \dots + |w_p|^{2p/r} < 0 \right\}$$

where p is a positive integer and r is a real number such that $0 < r \leq p$. For a function $f \in A^2(\Omega, (-\rho)^r)$, we let $F(z, w) = f(z)$ be the trivial extension of f to Ω_r^p . It easily follows from an iterated integral argument that $F \in A^2(\Omega_r^p)$.

The following lemma gives a relationship between the Bergman kernels of the inflated domain and base.

Lemma 1. *Using the notation above*

$$K_\Omega^r(z, \xi) = c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0)$$

where $c_{p,r} = \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w})$ and $K_\Omega^r(z, \xi)$ is the weighted Bergman kernel of Ω with weight $(-\rho)^r$.

Proof. We will follow a standard inflation argument (see for instance [FR75, Lig89]). Since Ω_r^p is a Hartogs domain with base Ω , the Bergman kernel of Ω_r^p can be written as

$$K_{\Omega_r^p}(z, w; \xi, \eta) = K_{\Omega_r^p}(z, 0; \xi, 0) + \sum_{|J| \geq 1} K_J(z, \xi) w^J \bar{\eta}^J$$

where J is a multiindex with nonnegative entries. Then for $f \in A^2(\Omega, (-\rho)^r)$ and $z \in \Omega$ we have (F below is the trivial extension of f)

$$(2) \quad f(z) = \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta) dV(\xi, \eta) + \sum_{|J| \geq 1} \int_{\Omega_r^p} K_J(z, \xi) w^J \bar{\eta}^J F(\xi, \eta) dV(\xi, \eta).$$

However, the integrals under the sum on the right hand side above all vanish.

Using the change of variables $\tilde{w}_j = \frac{w_j}{(-\rho(z))^{r/2p}}$ one can compute that

$$(3) \quad \int_{|w_1|^{2p/r} + \dots + |w_p|^{2p/r} < -\rho(z)} dV(w) = (-\rho(z))^r \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).$$

We denote

$$(4) \quad c_{p,r} = \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).$$

Then using (2),(3), and (4) we get

$$f(z) = \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta) dV(\xi, \eta) = c_{p,r} \int_{\Omega} K_{\Omega_r^p}(z, 0; \xi, 0) f(\xi) (-\rho(\xi))^r dV(\xi).$$

Therefore, $c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0) = K_{\Omega}^r(z, \xi)$. □

For a C^2 -smooth function ρ around a point $P \in \mathbb{C}^n$, $X = (x_1, \dots, x_n) \in \mathbb{C}^n$, and $Y = (y_1, \dots, y_n) \in \mathbb{C}^n$, we define the complex Hessian of ρ at P as

$$H_{\rho}(P; X, Y) = \sum_{j,k=1}^n \frac{\partial^2 \rho(P)}{\partial z_j \partial \bar{z}_k} x_j \bar{y}_k.$$

Furthermore, we use the notation $H_{\rho}(P; X) = H_{\rho}(P; X, X)$.

Lemma 2. *Assume that $z_0 \in b\Omega$ is a strongly pseudoconvex point. Then there exists $s > 0$ such that $(z, w) \in b\Omega_r^p$ is strongly pseudoconvex for $|z - z_0| < s$ and $w \neq 0$.*

Proof. Let $\tilde{\rho}(z, w) = \rho(z) + |w_1|^{2p/r} + \dots + |w_p|^{2p/r}$. Then $\tilde{\rho}$ is a C^2 -smooth function. Assume that $Q = (z, w) \in b\Omega_r^p$ is near z_0 and X is a complex tangential vector to $b\Omega_r^p$ at Q . Then X can be written as $X = X_n + X_p$ where X_n and X_p are the components of X in the z and w variables, respectively. Then

$$H_{\tilde{\rho}}(Q; X) = H_{\rho}(z; X_n) + H_{\tilde{\rho}}(Q; X_n, X_p) + H_{\tilde{\rho}}(Q; X_p, X_n) + H_{\tilde{\rho}}(Q; X_p).$$

However, $H_{\tilde{\rho}}(Q; X_n, X_p) = H_{\tilde{\rho}}(Q; X_p, X_n) = 0$ as z and w are decoupled in $\tilde{\rho}$. Then

$$H_{\tilde{\rho}}(Q; X) = H_{\rho}(z; X_n) + H_{\tilde{\rho}}(Q; X_p).$$

Let π denote the projection from a neighborhood of $b\Omega$ in \mathbb{C}^n onto $b\Omega$. Then $X_n = X_t + X_v$ where X_t is a tangential vector to $b\Omega$ at πz and X_v is a vector complex normal to $b\Omega$ at πz . Then

$$H_{\rho}(z; X_n) = H_{\rho}(z; X_t) + H_{\rho}(z; X_t, X_v) + H_{\rho}(z; X_v, X_t) + H_{\rho}(z; X_v).$$

We note that the complex Hessian H_{ρ} changes continuously and $w \rightarrow 0$ as $z \rightarrow z_0$ (here we assume that $(z, w) \in b\Omega_r^p$). Furthermore, $X_v \rightarrow 0$ as $z \rightarrow z_0$ (as the complex normal to $b\Omega$ at z_0 is parallel to the complex normal to $b\Omega_r^p$ at $(z_0, 0)$). Then, using the fact that z_0 is a strongly pseudoconvex point, we conclude that there exists $s > 0$ so that

$$H_{\rho}(z; X_n) \geq \frac{H_{\rho}(\pi z; X_t)}{2} > 0$$

for $|z - z_0| < s$ and $X_t \neq 0$. Also $H_{\tilde{\rho}}(Q; X_p) > 0$ whenever $X_p \neq 0$ and $w \neq 0$. Therefore, $H_{\tilde{\rho}}(Q; X) > 0$ for $Q = (z, w) \in b\Omega_r^p$ such that $|z - z_0| < s$ and $w \neq 0$. \square

Next we will prove some statements about compactness of single Toeplitz and Hankel operators.

Lemma 3. *Let $\phi \in L^\infty(\Omega)$, $\{f_j\}$ be a bounded sequence in $A^2(\Omega, (-\rho)^r)$ and F_j be the trivial extension of f_j to Ω_r^p for each j . Assume that $\{H_\phi^{\Omega_r^p} F_j\}$ is convergent in $L^2(\Omega_r^p)$. Then $\{H_\phi^{\Omega, r} f_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$.*

Proof. We will abuse the notation and denote the trivial extension of ϕ to Ω_r^p by ϕ . We assume that $\{H_\phi^{\Omega_r^p} F_j\}$ is convergent (and hence Cauchy). Let

$$G_j(z, w) = (H_\phi^{\Omega_r^p} F_j)(z, w)$$

and $g_j(z) = G_j(z, 0)$. Then G_j is holomorphic in w because

$$\frac{\partial G_j}{\partial \bar{w}_k} = \frac{\partial}{\partial \bar{w}_k} (I - P^{\Omega_r^p})(F_j \phi) = \frac{\partial (F_j \phi)}{\partial \bar{w}_k} = 0$$

for all j and $1 \leq k \leq p$. We note that $\frac{\partial (F_j \phi)}{\partial \bar{w}_k} = 0$ as $F_j \phi$ is independent of w_k . Then $|G_j(z, w) - G_k(z, w)|^2$ is subharmonic in w and using the mean value property for subharmonic functions together with (3) and (4) one can show that

$$|g_j(z) - g_k(z)|^2 \leq \frac{1}{c_{p,r}(-\rho(z))^r} \int_{|w_1|^{2p/r} + \dots + |w_p|^{2p/r} < -\rho(z)} |G_j(z, w) - G_k(z, w)|^2 dV(w)$$

for $j = 1, 2, \dots$ and $z \in \Omega$. By integrating over Ω we get

$$c_{p,r} \|g_j - g_k\|_{L^2_{(0,1)}(\Omega, (-\rho)^r)}^2 \leq \|G_j - G_k\|_{L^2_{(0,1)}(\Omega_r^p)}^2$$

for $j, k = 1, 2, \dots$. Then $\{g_j\}$ is a Cauchy sequence in $L^2_{(0,1)}(\Omega, (-\rho)^r)$ (and hence convergent) because $\|G_j - G_k\|_{L^2_{(0,1)}(\Omega_r^p)} \rightarrow 0$ as $j, k \rightarrow \infty$.

Let $h_j(z) = P^{\Omega_r^p}(\phi F_j)(z, 0)$. Then

$$c_{r,p} \|h_j\|_{L^2(\Omega, (-\rho)^r)}^2 \leq \|P^{\Omega_r^p}(\phi F_j)\|_{L^2(\Omega_r^p)}^2 \leq \|\phi F_j\|_{L^2(\Omega_r^p)}^2 = c_{r,p} \|\phi f_j\|_{L^2(\Omega, (-\rho)^r)}^2 < \infty$$

for each j . Hence, $h_j \in A^2(\Omega, (-\rho)^r)$ and $(I - P^{\Omega, r})h_j = 0$ for all j . We get equality between the last terms above because F_j and ϕ are independent of w . Now

$$\begin{aligned} (I - P^{\Omega, r})g_j &= (I - P^{\Omega, r}) \left(\phi f_j - P^{\Omega_r^p}(\phi F_j)(\cdot, 0) \right) \\ &= (I - P^{\Omega, r})(\phi f_j) - (I - P^{\Omega, r})(h_j) \\ &= H_\phi^{\Omega, r} f_j. \end{aligned}$$

Therefore, the sequence $\{H_\phi^{\Omega, r} f_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$. \square

Lemma 4. *Let r be a nonnegative real number and Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ . Assume that $\phi \in C(\overline{\Omega})$ such that $\phi(z) = 0$ if z is a strongly pseudoconvex point in $b\Omega$. Then T_ϕ^r is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$.*

Proof. Let $\{f_j\}$ be a sequence in $A^2(\Omega, (-\rho)^r)$ that (without loss of generality) converges to 0 weakly about strongly pseudoconvex points. Then $f_j \rightarrow 0$ weakly as $j \rightarrow \infty$ and there is a neighborhood U of weakly pseudoconvex points in $b\Omega$ such that

$$\|f_j\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Using the uniform boundedness principle and the fact that $f_j \rightarrow 0$ weakly we conclude that the sequence $\{f_j\}$ is bounded in $A^2(\Omega, (-\rho)^r)$. Furthermore, Cauchy estimates together with Montel's Theorem (and the fact that $f_j \rightarrow 0$ weakly) imply that $\{f_j\}$ converges to zero uniformly on compact subsets of Ω . Using the fact that $\phi = 0$ on strongly pseudoconvex points, one can show that $\phi f_j \rightarrow 0$ in $A^2(\Omega, (-\rho)^r)$. Therefore, $T_\phi^r f_j \rightarrow 0$ in $A^2(\Omega, (-\rho)^r)$. That is, T_ϕ^r is compact about strongly pseudoconvex points on in $A^2(\Omega, (-\rho)^r)$. \square

Let Ω be a domain in \mathbb{C}^n . Then $z \in b\Omega$ is said to have a holomorphic (plurisubharmonic) peak function if there exists a holomorphic (plurisubharmonic) f that is continuous on $\overline{\Omega}$ such that $f(z) = 1$ and $|f(w)| < 1$ ($f(w) < 1$) for $w \in \overline{\Omega} \setminus \{z\}$.

Next we show that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

Proposition 2. *Let r be a nonnegative real number, Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ , and $\phi \in C(\overline{\Omega})$. Then $H_\phi^r : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$ is compact about strongly pseudoconvex points.*

Proof. We will prove more (see Corollary 2 below). First of all, for any $\phi \in C(\overline{\Omega})$ there exists $\{\phi_j\} \subset C^1(\overline{\Omega})$ such that $\phi_j \rightarrow \phi$ uniformly on $\overline{\Omega}$ as $j \rightarrow \infty$. Furthermore, one can easily show that if $H_{\phi_j}^r$ is compact for all j then H_ϕ^r is compact. Therefore, for the rest of the proof we will assume that $\phi \in C^1(\overline{\Omega})$. Secondly, the proof for $r = 0$ does not require the inflation argument in the next paragraph and hence it is easier than the case $r > 0$. Since both proofs are similar, except for the inflation argument, in the rest of the proof, we will assume that $r > 0$.

Let $z_0 \in b\Omega$ be a strongly pseudoconvex point. Let $\varepsilon > 0$ be such that $X_0 = b\Omega \cap \overline{B(z_0, \varepsilon)}$ consists of strongly pseudoconvex points. Let us define $X_j = b\Omega_r^p \cap (\overline{B(z_0, \varepsilon)} \setminus \{|w| < 1/j\})$ for $j = 1, 2, 3, \dots$. Then X_0 is B-regular as any point in X_0 has a holomorphic (hence plurisubharmonic) peak function on $\Omega \subset \mathbb{C}^n$ and the same function is also a plurisubharmonic peak function on $\Omega_r^p \subset \mathbb{C}^{n+p}$. Lemma 2 implies that we can shrink ε , if necessary, so that X_j 's are composed of strongly pseudoconvex points. Hence, they are B-regular. Then [Sib87, Proposition 1.9] implies that $b(\Omega_r^p \cap B(z_0, \varepsilon)) \subset (\cup_{j=0}^\infty X_j) \cup bB(z_0, \varepsilon)$ is B-regular (satisfies Property (P) in Catlin's terminology) and, hence, the $\bar{\partial}$ -Neumann operator on $\Omega_r^p \cap B(z_0, \varepsilon)$ is compact (see [Str10, Theorem 4.8])

and [Cat84]). Then $H_\phi^{\Omega_r^p \cap B(z_0, \varepsilon)}$ is compact (see [Str10, Proposition 4.1]) and Lemma 3 implies that $H_\phi^{\Omega \cap B(z_0, \varepsilon), r}$ is compact.

Next we will use local compact solution operators to show that H_ϕ^r is compact about strongly pseudoconvex points. Let $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$ be a sequence weakly convergent about strongly pseudoconvex points. Then there exists an open neighborhood U of the set of weakly pseudoconvex points in $b\Omega$ such that

- i. $\{f_j\}$ is weakly convergent,
- ii. $\|f_j - f_k\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0$ as $j, k \rightarrow \infty$.

Let us choose $\{p_k : k = 1, \dots, m\} \subset b\Omega \setminus U$ and $\varepsilon > 0$ such that

- i. $b\Omega \setminus U \subset \bigcup_{k=1}^m B(p_k, \varepsilon)$
- ii. $H_\phi^{k, r} = H_\phi^{B(p_k, \varepsilon) \cap \Omega, r}$ is compact on $A^2(B(p_k, \varepsilon) \cap \Omega, (-\rho)^r)$ for $k = 1, \dots, m$.

Let us choose a strongly pseudoconvex domain $\Omega_{-1} \Subset \Omega$ and smooth cut-off functions $\chi_{-1} \in C_0^\infty(\Omega_{-1})$, $\chi_0 \in C_0^\infty(U)$, and $\chi_k \in C_0^\infty(B(p_k, \varepsilon))$ for $k = 1, \dots, m$ such that $\sum_{k=-1}^m \chi_k \equiv 1$ on $\overline{\Omega}$.

Let $H_\phi^{-1, r} = H_\phi^{\Omega_{-1}, r}$, $H_\phi^{0, r} = H_\phi^{U \cap \Omega, r}$, and $g_j = \sum_{k=-1}^m \chi_k H_\phi^{k, r} f_j$. We note that $H_\phi^{-1, r}$ is compact as $\Omega_{-1} \Subset \Omega$ is strongly pseudoconvex; $\{H_\phi^{0, r} f_j\}$ is convergent as $\{f_j\}$ is convergent in $A^2(U \cap \Omega, (-\rho)^r)$; and by the second paragraph of this proof, $\{H_\phi^{k, r}\}$ is compact for each $k = 1, \dots, m$. Therefore, $\{g_j\}$ is convergent in $A^2(\Omega, (-\rho)^r)$. Furthermore,

$$\bar{\partial} g_j = f_j \bar{\partial} \phi + \sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k, r} f_j.$$

Then $\left\{ \sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k, r} f_j \right\}$ is a convergent sequence of $\bar{\partial}$ -closed $(0, 1)$ -forms as both $\bar{\partial} g_j$ and $f_j \bar{\partial} \phi$ are $\bar{\partial}$ -closed. Let $Z^r : L_{(0,1)}^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$ be a bounded linear solution operator to $\bar{\partial}$ (see [Hör65]). Let

$$h_j = g_j - Z^r \sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k, r} f_j.$$

Then $\{h_j\}$ is convergent and $\bar{\partial} h_j = f_j \bar{\partial} \phi$. So by taking projection on the orthogonal complement of $A^2(\Omega, (-\rho)^r)$ we get $(I - P^r)h_j = H_\phi^r f_j$. Therefore, $\{H_\phi^r f_j\}$ is convergent. \square

We note that the first part of the proof of Proposition 2 implies that if Ω satisfies property (P) so does Ω_r^p . Then we have the following corollary.

Corollary 2. *Let r be a nonnegative real number and Ω be a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^n with a defining function ρ . Assume that Ω satisfies property (P) of Catlin (or B-regularity of Sibony). Then*

- i. $\bar{\partial}$ has a compact solution operator on $K_{(0,1)}^2(\Omega, (-\rho)^r)$, the weighted $\bar{\partial}$ -closed $(0, 1)$ -forms,
- ii. $H_\phi^r : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$ is compact for all $\phi \in C(\overline{\Omega})$.

Proof. We will prove i. only as the proof of ii. is similar. Let $\{f_j\}$ be a bounded sequence in $K_{(0,1)}^2(\Omega, (-\rho)^r)$. Then $\{F_j\}$ is a bounded sequence in $K_{(0,1)}^2(\Omega_r^p)$. Since Ω satisfies property (P) so does Ω_r^p . Then $\{\bar{\partial}^* N^{\Omega_r^p} F_j\}$ has a convergent subsequence in $L^2(\Omega_r^p)$ where $N^{\Omega_r^p}$ is the $\bar{\partial}$ -Neumann operator on $L_{(0,1)}^2(\Omega_r^p)$. Furthermore, $\bar{\partial}^* N^{\Omega_r^p} F_j(\cdot, 0) \in L^2(\Omega, (-\rho)^r)$ (as $\bar{\partial}^* N^{\Omega_r^p} F_j$ is holomorphic in w), $\bar{\partial} \bar{\partial}^* N^{\Omega_r^p} F_j(\cdot, 0) = f_j$ for all j , and $\{\bar{\partial}^* N^{\Omega_r^p} F_j(\cdot, 0)\}$ has a convergent subsequence in $L^2(\Omega, (-\rho)^r)$. Therefore, $\bar{\partial}$ has a compact solution operator $R \bar{\partial}^* N^{\Omega_r^p} E$ on $K_{(0,1)}^2(\Omega, (-\rho)^r)$ where E is the trivial extension operator and R is the restriction from Ω_r^p onto Ω . \square

The following Lemma is essentially contained in the proof of [AE01, Proposition 1.3]. We present it here for the convenience of the reader.

Lemma 5. *Let r be a nonnegative real number, Ω be bounded domain in \mathbb{C}^n , and $\phi \in C(\bar{\Omega})$. Assume that $z_0 \in b\Omega$ has a holomorphic peak function. Then*

$$\lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0).$$

Proof. First, we prove that for any neighborhood U of z_0

$$(5) \quad \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \rightarrow 0 \text{ as } z \rightarrow z_0.$$

Indeed, for given U and $\varepsilon > 0$ first we choose a holomorphic peak function g such that $|g(w)| \leq \varepsilon$ for all $w \in \Omega \setminus U$. This can be simply done by taking a high enough power of the holomorphic peak function g . Then we choose $\delta > 0$ such that if $|z - z_0| < \delta$ and $z \in \Omega$ then $|g(z)| > 1 - \varepsilon$. In this case,

$$\begin{aligned} \int_U |k_z^r(w)|^2 (-\rho(w))^r dV(w) &\geq \int_U |g(w)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq \left| \int_\Omega g(w) |k_z^r(w)|^2 (-\rho(w))^r dV(w) \right| \\ &\quad - \left| \int_{\Omega \setminus U} g(w) |k_z^r(w)|^2 (-\rho(w))^r dV(w) \right| \\ &\geq |g(z)| - \int_{\Omega \setminus U} |g(w)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq 1 - \varepsilon - \varepsilon \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq 1 - 2\varepsilon \end{aligned}$$

whenever $|z - z_0| < \delta$. This implies that for a given neighborhood U and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \leq \varepsilon.$$

This gives (5).

Now for $\varepsilon > 0$, we choose a neighborhood U of z such that $|\phi(w) - \phi(z_0)| \leq \varepsilon$ for all $w \in U$. Then for this neighborhood U and the same ε we choose $\delta > 0$ such that if $|z - z_0| < \delta$ then $\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \leq \frac{\varepsilon}{1+2\|\phi\|_{L^\infty}}$. In this case,

$$\begin{aligned} \left| B_r T_\phi^r(z) - \phi(z_0) \right| &\leq \int_{\Omega} |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &= \int_U |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\quad + \int_{\Omega \setminus U} |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\leq \varepsilon \int_U |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\quad + 2\|\phi\|_{L^\infty} \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This indeed concludes $\lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0)$. □

We note that on any bounded domain, we have (see [ČŠ14, Lemma 1])

$$T_{\phi_2}^r T_{\phi_1}^r = T_{\phi_2 \phi_1}^r - H_{\phi_2}^{r*} H_{\phi_1}^r.$$

Using the fact above inductively one can prove the following lemma.

Lemma 6. *Let r be a nonnegative real number and Ω be a C^1 -smooth bounded domain in \mathbb{C}^n with a defining function ρ . Supposed $\phi_1, \dots, \phi_m \in L^\infty(\Omega)$. Then*

$$\begin{aligned} T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_2}^r T_{\phi_1}^r &= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r - T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_3}^r H_{\phi_2}^{r*} H_{\phi_1}^r \\ &\quad - T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_4}^r H_{\phi_3}^{r*} H_{\phi_2 \phi_1}^r - \cdots - H_{\phi_m}^{r*} H_{\phi_{m-1} \cdots \phi_2 \phi_1}^r \\ &= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r + S^r \end{aligned}$$

where S^r is a finite sum of finite products of operators and each product starts with a Hankel operator.

Therefore, if the symbols ϕ_1, \dots, ϕ_m are continuous on $\overline{\Omega}$ we can write

$$(6) \quad T_\phi^r \cdots T_{\phi_m}^r = T_{\phi_1 \cdots \phi_m}^r + S^r$$

where S^r is a finite sum of finite products of operators such that each product starts with a Hankel operator with symbol continuous on $\overline{\Omega}$.

Let Ω be a pseudoconvex domain in \mathbb{C}^n and $z_0 \in b\Omega$. Then we call z_0 a *bumping point* if for any $\delta > 0$ there exists a pseudoconvex domain Ω_1 such that $\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, \delta)$.

Lemma 7. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n with Lipschitz boundary and $z_0 \in b\Omega$ be a bumping point. Then $k_z \rightarrow 0$ weakly as $z \rightarrow z_0$.*

Proof. Let us fix $f \in A^2(\Omega)$ and choose $r_1, r_2 > 0$ so that $0 < r_1 < r_2$ and the outward unit vector ν is transversal to $B(z_0, 2r_2) \cap b\Omega$. Since z_0 is a bumping point we choose a bounded pseudoconvex domain Ω_1 such that

$$\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, r_1).$$

So while Ω_1 contains a small neighborhood of z_0 we have $\Omega \setminus B(z_0, r_1) = \Omega_1 \setminus B(z_0, r_1)$.

Let us define $g_\varepsilon = (1 - \chi)f + \chi f_\varepsilon$ where $f_\varepsilon(z) = f(z - \varepsilon\nu)$ and $\chi \in C_0^\infty(B(z_0, r_2))$ such that $\chi \equiv 1$ on a neighborhood of $\overline{B(z_0, r_1)}$. Then

- i. $f_\varepsilon \in A^2(\Omega \cap B(z_0, r_2))$ and $f_\varepsilon \rightarrow f$ in $L^2(\Omega \cap B(z_0, r_2))$,
- ii. $g_\varepsilon|_{\Omega \cap B(z_0, r_2)}$ is C^∞ -smooth and $g_\varepsilon \rightarrow f$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$.

Furthermore, $\bar{\partial}g_\varepsilon$ is a $\bar{\partial}$ -closed $(0, 1)$ -form on Ω_1 ($\bar{\partial}g_\varepsilon$ is well defined on Ω_1 as $\bar{\partial}\chi = 0$ on $B(z_0, r_1)$) for all small $\varepsilon > 0$ and

$$\|\bar{\partial}g_\varepsilon\|_{L^2(\Omega_1)} \leq \|f - f_\varepsilon\|_{L^2(\Omega \cap B(z_0, r_2))} \|\bar{\partial}\chi\|_{L^\infty(B(z_0, r_2))} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then Hörmander's theorem [Hör65] implies that there exist a constant $c_{\Omega_1} > 0$ (depending on Ω_1) and $h_\varepsilon \in L^2(\Omega_1)$ such that $\bar{\partial}h_\varepsilon = \bar{\partial}g_\varepsilon$ and $\|h_\varepsilon\|_{L^2(\Omega_1)} \leq c_{\Omega_1} \|\bar{\partial}g_\varepsilon\|_{L^2(\Omega_1)}$. Furthermore, since $\bar{\partial}$ is elliptic on the interior and $\bar{\partial}g_\varepsilon$ is C^∞ -smooth on Ω_1 , we have $h_\varepsilon \in C^\infty(\Omega_1)$.

We define $f_n = g_{1/n} - h_{1/n}$. Then we have

- i. $f_n \in A^2(\Omega)$ and $f_n \rightarrow f$ in $A^2(\Omega)$,
- ii. $f_n|_{\Omega \cap B(z_0, r_1)} \in C^\infty(\overline{\Omega \cap B(z_0, r_1)})$.

So $\{f_n\}$ is a sequence converging to f and each member of the sequence is smooth up to the boundary of Ω on a neighborhood of z_0 .

Finally, we will show weak convergence of k_z to 0 as $z \rightarrow z_0$.

$$|\langle f, k_z \rangle| \leq |\langle f - f_n, k_z \rangle| + |\langle f_n, k_z \rangle| \leq \|f - f_n\|_{L^2(\Omega)} + \frac{|f_n(z)|}{\sqrt{K(z, z)}}.$$

The first term on the right hand side can be made arbitrarily small for large enough n , because $\|f - f_n\|_{L^2(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. So for $\delta > 0$ given we choose n_δ so that $\|f - f_{n_\delta}\|_{L^2(\Omega)} \leq \delta$. Then since f_{n_δ} is C^∞ -smooth on $\overline{\Omega \cap B(z_0, r_1)}$ (and $K(z, z) \rightarrow \infty$ as $z \rightarrow z_0$) we conclude that $|f_{n_\delta}(z)|/\sqrt{K(z, z)} \rightarrow 0$ as $z \rightarrow z_0$. Hence, $\limsup_{z \rightarrow z_0} |\langle f, k_z \rangle| \leq \delta$ for arbitrary $\delta > 0$. Therefore, $k_z \rightarrow 0$ weakly as $z \rightarrow z_0$. \square

We believe that the following lemma is of interest in its own right.

Lemma 8. Assume that $z_0 \in b\Omega$ is a strongly pseudoconvex point. Then $(z_0, 0) \in b\Omega_r^p$ is a bumping point for Ω_r^p where p is a positive integer and r is a positive real number such that $p \geq r$.

Proof. For a given $\delta > 0$ let $\chi_1 \in C_0^\infty(\{z \in \mathbb{C}^n : |z - z_0| < \delta/4\})$ and $\chi_2 \in C_0^\infty(\{w \in \mathbb{C}^p : |w| < \delta/4\})$ such that $\chi_1 \equiv 1$ near z_0 while $\chi_2 \equiv 1$ near 0. We choose δ small enough so that all the points in $b\Omega \cap B(z_0, \delta)$ are strongly pseudoconvex. Let us define

$$\rho_\varepsilon(z, w) = \rho(z) + |w_1|^{2p/r} + \cdots + |w_p|^{2p/r} - \varepsilon\chi_1(z)\chi_2(w)$$

and $\tilde{\Omega}_\varepsilon = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : \rho_\varepsilon(z, w) < 0\}$. For sufficiently small $\varepsilon > 0$ one can check that the Levi form of $\tilde{\Omega}_\varepsilon$ is positive semidefinite. Furthermore, $\tilde{\Omega}_\varepsilon \setminus B((z_0, 0), \delta) = \Omega_r^p \setminus B((z_0, 0), \delta)$ and $\rho_\varepsilon(z_0, 0) = -\varepsilon < 0$. Hence,

$$\{(z_0, 0)\} \cup \Omega_r^p \subset \tilde{\Omega}_\varepsilon \subset \Omega_r^p \cup B((z_0, 0), \delta).$$

That is, $(z_0, 0)$ is a bumping point for Ω_r^p . □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. In case $r = 0$, the proof of the theorem simplifies greatly as inflation and the related techniques are unnecessary. So we will prove the more difficult case, $r > 0$.

First we assume that T is compact about strongly pseudoconvex points. Let Ω_r^p be defined as in (1). Then Ω_r^p is a C^2 -smooth bounded pseudoconvex domain in \mathbb{C}^{n+p} . Let $z_0 \in b\Omega$ be a strongly pseudoconvex point and $K_{\Omega_r^p}(\xi, \xi'; z, z')$ be the Bergman kernel for Ω_r^p . Then Lemma 1 implies that

$$k_z^r(\xi) = \sqrt{c_{p,r}} \frac{K_{\Omega_r^p}(\xi, 0; z, 0)}{\sqrt{K_{\Omega_r^p}(z, 0; z, 0)}}$$

where $c_{p,r} = \int_{|w_1|^{2p/r} + \dots + |w_p|^{2p/r} < 1} dV(w)$. Furthermore, Lemma 7 and Lemma 8 imply that

$$\left\langle \frac{K_{\Omega_r^p}(\cdot, \cdot; z, 0)}{\sqrt{K_{\Omega_r^p}(z, 0; z, 0)}}, F \right\rangle_{A^2(\Omega_r^p)} \longrightarrow 0 \text{ as } z \rightarrow z_0 \text{ for all } F \in A^2(\Omega_r^p).$$

Then

$$(7) \quad \langle k_z^r, f \rangle_{A^2(\Omega, (-\rho)^r)} = \sqrt{c_{p,r}} \left\langle \frac{K_{\Omega_r^p}(\cdot, \cdot; z, 0)}{\sqrt{K_{\Omega_r^p}(z, 0; z, 0)}}, F \right\rangle_{A^2(\Omega_r^p)} \rightarrow 0 \text{ as } z \rightarrow z_0$$

for all $f \in A^2(\Omega, (-\rho)^r)$. Hence, $k_z^r \rightarrow 0$ weakly in $A^2(\Omega, (-\rho)^r)$ as $z \rightarrow z_0$. Furthermore, there exists an open neighborhood U of z_0 such that weak pseudoconvex points are contained in $b\Omega \setminus \overline{U}$; and, as in the proof of (5), one can show that

$$\|k_z^r\|_{L^2(\Omega \setminus \overline{U}, (-\rho)^r)} \rightarrow 0 \text{ as } z \rightarrow z_0.$$

Therefore, $\{k_z^r\}$ converges to 0 weakly about strongly pseudoconvex points as $z \rightarrow z_0$.

Since T is compact about strongly pseudoconvex points and such operators map sequences of holomorphic functions weakly convergent about strongly pseudoconvex points to convergent sequences; we conclude that

$$B_r T(z) = \langle T k_z^r, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)} \rightarrow 0$$

as $z \rightarrow z_0$.

Next we prove the other direction. As a first step we assume that T is a finite sum of finite products of Toeplitz operators on $A^2(\Omega_r^p)$ with symbols continuous on $\overline{\Omega}$. Furthermore, we assume

that

$$\lim_{z \rightarrow z_0} B_r T(z) = 0$$

for any strongly pseudoconvex point $z_0 \in b\Omega$.

Lemma 6 implies that

$$(8) \quad T = T_\phi^r + S^r$$

where $\phi \in C(\overline{\Omega})$ and S^r a sum of operators that start with a Hankel operator with a symbol continuous on $\overline{\Omega}$.

Lemma 5 implies that

$$(9) \quad \lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0)$$

and Proposition 2 implies that S^r is compact about strongly pseudoconvex points. Furthermore, Lemma 3 together with the fact that $k_z^r \rightarrow 0$ weakly as $z \rightarrow z_0$ (see (7)) imply that $H_\psi^r k_z^r \rightarrow 0$ for any $\psi \in C(\overline{\Omega})$. Hence, $B_r S^r(z) \rightarrow 0$ as $z \rightarrow z_0$. Combining this with (8) and (9) we can conclude that

$$\phi(z_0) = \lim_{z \rightarrow z_0} B_r T(z) = 0.$$

Since z_0 was an arbitrary strongly pseudoconvex point, we have $\phi = 0$ on all the strongly pseudoconvex boundary points. Then Lemma 4 and the fact that S^r is compact about strongly pseudoconvex points imply that T is compact about strongly pseudoconvex points.

Finally, we assume that $T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r)$. Then, using Lemma 6, for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in C(\overline{\Omega})$ and an operator S_ε^r , compact about strongly pseudoconvex points, such that

$$\|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| \leq \varepsilon.$$

Then for $z \in \Omega$ we have

$$\begin{aligned} \left| B_r T(z) + B_r T_{\phi_\varepsilon}^r(z) + B_r S_\varepsilon^r(z) \right| &= \left| \langle T k_z^r + T_{\phi_\varepsilon}^r k_z^r + S_\varepsilon^r k_z^r, k_z^r \rangle_r \right| \\ &\leq \|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| \\ &\leq \varepsilon. \end{aligned}$$

Since $B_r S_\varepsilon^r(z) \rightarrow 0$ and $B_r T_{\phi_\varepsilon}^r(z) \rightarrow \phi_\varepsilon(z_0)$ (and we assume that $B_r T(z) \rightarrow 0$ as $z \rightarrow z_0$) as $z \rightarrow z_0$ we have $|\phi_\varepsilon(z_0)| \leq \varepsilon$. That is, $|\phi_\varepsilon| \leq \varepsilon$ on strongly pseudoconvex points of Ω . We choose $\psi_\varepsilon \in C(\overline{\Omega})$ such that $\psi_\varepsilon = 0$ on strongly pseudoconvex boundary points of Ω and

$$\sup\{|\psi_\varepsilon(z) - \phi_\varepsilon(z)| : z \in \overline{\Omega}\} \leq 2\varepsilon.$$

Then one can show that $T_{\psi_\varepsilon}^r$ is compact about strongly pseudoconvex points and

$$\|T_{\phi_\varepsilon}^r - T_{\psi_\varepsilon}^r\| \leq 2\varepsilon.$$

Then

$$\|T + T_{\psi_\varepsilon}^r + S_\varepsilon^r\| \leq \|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| + \|T_{\psi_\varepsilon}^r - T_{\phi_\varepsilon}^r\| \leq 3\varepsilon.$$

Therefore, T is in the norm closure of the compact about strongly pseudoconvex points operators. Finally, Proposition 1 implies that T is compact about strongly pseudoconvex points. \square

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